Topics on Technology and Profit Maximization

Lecture1; V, chs. 1 and 2; MWG, ch. 5

1. Technology

1.1 Describing technologies

- The firm's production decisions can be represented as a "netput" vector **y** = $\{y_1, y_2, \ldots, y_n\}$. The set of technologically feasible production plans is called the *production possibility set* and is denoted Y.
- If we only have one output the *production function* $f(\mathbf{x})$ describes the maximum level of output that can be obtained for a given vector of inputs **x**. It constitutes the border of the production possibility set. In the case of several outputs the production set can be described by a *transformation function* T(**y**) implicitly defined by $Y=\{\mathbf{y}\in\mathbb{R}^n: T(\mathbf{y})\leq 0\}$ and such that T(**y**) = 0 only if the production plan is efficient. The set of efficient production plans is called the *transformation frontier*.
- The *input requirement set for* y, V(y), is the set of inputs **x** sufficient to produce y, i.e., $V(y)=\{\mathbf{x}\in\mathbb{R}^{n-1}: (y,-\mathbf{x}) \in Y\}$. Formally, input combinations **x** that yield the same (maximum) level of output y, define an *isoquant* $Q(y)=\{\mathbf{x}\in\mathbb{R}^{n-1}: \mathbf{x}\in V(y) \text{ and } \mathbf{x}\notin V(y'), y'>y\}$.

Example: Cobb-Douglas, V, p.4.

1.2 Properties of technologies

- Possibility of inaction: $\mathbf{0}\in Y$;
- Free disposal: $\mathbf{y}\in Y$, $\mathbf{y}'\leq \mathbf{y}\Rightarrow \mathbf{y}'\in Y$;
- Convexity: $\mathbf{y}, \mathbf{y}'\in Y$, $t\mathbf{y}+(1-t)\mathbf{y}'\in Y$, for all $0\leq t\leq 1$;
- No free lunch: $\mathbf{y}\in Y$ and $\mathbf{y}\geq\mathbf{0}\Rightarrow \mathbf{y}=\mathbf{0}$;
- Additivity: $\mathbf{y}, \mathbf{y}'\in Y\Rightarrow \mathbf{y}+\mathbf{y}'\in Y$;
- Irreversibility: $\mathbf{y}\in Y$ and $\mathbf{y}\neq\mathbf{0}\Rightarrow -\mathbf{y}\notin Y$;
- Nonincreasing returns to scale: $\mathbf{y}\in Y\Rightarrow t\mathbf{y}\in Y$, for all $0\leq t\leq 1$;
- Nondecreasing returns to scale: $\mathbf{y}\in Y\Rightarrow t\mathbf{y}\in Y$, for all $1\leq t$;
- Constant returns to scale (CRS): $\mathbf{y}\in Y\Rightarrow t\mathbf{y}\in Y$, for all $0\leq t$;

1.3 Properties of technologies cont.

The *technical rate of substitution* (TRS) between two inputs is simply the slope of Q(y) and can be obtained by totally differentiating $f(\mathbf{x})$ = y. It is thus the ratio of the marginal products of the inputs in question: $\dfrac{dx_1}{dx_2} = -\dfrac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}$.

The *elasticity of substitution* σ tells us how much the ratio of inputs, change with a change in the *TRS,* along the isoquant. If the isoquant is very curved then a considerable change in the slope will only be associated with a minor change in the ratio of inputs: $\sigma = \dfrac{d\ln(\frac{x_1}{x_2})}{d\ln|TRS|}$.

There are two classes of production functions with the convenient property that the *TRS*s are independent of the scale of production (i.e., a proportional change of all inputs starting from any point on a certain isoquant leads to the same change in output): *homogeneous functions* and *homothetic functions*. A function f(.) is *homogeneous of degree k* if $f(tx)=t^k f(x)$, for all t>0. To assume that a function is homothetic is a weaker assumption than to assume homogeneity but it still preserves the central property. A function g(.) is *homothetic* if g(x) = F(f(x)) where f(.) is homogenous of degree 1 and F'> 0.

Note that:

- A linear homogenous function (k = 1) has CRS for all x and has marginal products that are independent of the scale;
- A technology exhibits decreasing returns to scale iff $f(t\mathbf{x})<tf(\mathbf{x})$, for all t>1;
- A technology exhibits increasing returns to scale iff $f(t\mathbf{x})>tf(\mathbf{x})$, for all t>1.

Example: Cobb-Douglas, V, p.12 and 14.

2. Profit maximization

2.1 Profit maximization problem

The profit maximization problem for a firm facing given prices is to choose "netputs" to maximize profits. The maximum profit that can be obtained with a technologically feasible production plan for a given price vector **p** is given by the *profit function* Π. The firm thus solves $\Pi(p) = \underset{y}{Max}\, py \ s.t.\ y \in Y$ or, using the transformation function, $\Pi(p) = \underset{y}{Max}\, py \ s.t.\ T(y) = 0$. The FOC's for profit maximization imply $\dfrac{p_i}{pj} = \dfrac{\frac{\partial T}{\partial x_i}}{\frac{\partial T}{\partial xj}}, i, j = 1,..., n-1.$ The ratios of the partial derivatives of T describe the technological relationship between two variables. The interpretation depends on whether the goods in question are inputs or outputs: (a) if both netputs are inputs this corresponds to the technical rate of substitution TRS_{ij} and determines the optimal input mix; (b) if i is input and j an output this corresponds to MP_j and gives the optimal input level (c) If both netputs are outputs this corresponds to MRT_{ij} and gives the optimal output mix.

If the firm produces only one output, the problem can be written as $\Pi(p) = \underset{x}{Max}\, pf(x) - wx \ s.t.\ x \geq 0,$ where p denotes output price and w is a vector of input prices.

The FOC's for profit maximization in the latter case are $\frac{\partial f(x^*)}{\partial x_i} \le \frac{w_i}{p}, x_i^* \ge 0, (p\frac{\partial f(x^*)}{\partial x_i} - w_i)x^*_i = 0, i = 1,...,n-1$. In an interior solution, the marginal revenue product should equal the marginal cost.
The SOC for this case requires the Hessian matrix $\mathbf{D^2f(x)}$, which is a symmetric matrix, to be negative semi-definite at the optimum, i.e., $\mathbf{hD^2f(x^*)h}^t \le 0$, for all **h**.

Example: Cobb-Douglas, V, p.30.

2.2 Implications of profit maximization

a. Demand and supply functions

The *factor demand function* x(p,**w**) gives the optimal choice of inputs for each vector of prices (p,**w**); the function y(p,**w**)=f(x(p,**w**)) is called the supply function.

b. Comparative statics

Given a list of price vectors $\mathbf{p}^t$ and the associated optimal "netput" vectors $\mathbf{y}^t$, t=1,…,T, a necessary condition for profit maximization is that $\mathbf{p}^t\,\mathbf{y}^t \ge \mathbf{p}^t\,\mathbf{y}^s$ for all t,s=1,…,T. This is the Weak Axiom of Profit Maximization (WAPM) and it implies that $\triangle\mathbf{p}\triangle\mathbf{y} \ge 0$.